MA 123 - Elementary Calculus and Its Applications

Chapter 2 Homework

Due: Friday, June 14, 2013

1. Find the average rate of change of $f(x) = \sqrt{3x^2+1}$ as x changes from 1 to 4.

$$\underset{1\to 4}{\text{AROC}} = \frac{f(4)-f(1)}{4-1} = \frac{\sqrt{3(4)^2+1}-\sqrt{3(1)^2+1}}{4-1} = \frac{\sqrt{49}-\sqrt{4}}{4-1} = \frac{7-2}{4-1} = \boxed{\frac{5}{3}}$$

2. A charter bus full of U.K. students on their Spring Break drives from Lexington to Miami, FL. The cities are 900 miles apart, and the distance from Lexington at t hours after the bus leaves Lexington is given by $d(t) = 45t + t^2$. What is the average speed of the bus in miles per hour during the trip from Lexington to Miami?

(Hint: First find how long it takes for the bus to get from Lexington to Miami.)

We already know the total distance that the bus travels (900 miles), but we need to find the time elapsed during the trip. We have a function that tells us how far away the bus is from Lexington at any given moment, so set it equal to 900 and solve for t in order to find how long it takes for the bus to travel to Miami, that is, solve for t such that $900 = t^2 + 45t$. *So, we have* $0 = t^2 + 45t - 900 = (t-15)(t+60)$. *Thus,* $t = 15$ *or* $t = -60$. *Since time can never be negative, it must take the bus* 15 *hours to get to Miami. Then, the average velocity of the bus is* $\frac{900 \text{ miles}}{15 \text{ hours}} = \boxed{60 \text{ miles per hour}}$.

3. Let $g(x) = (x-1)^2$. Compute $\dfrac{g(3+h)-g(3)}{h}$.

$$\frac{g(3+h)-g(3)}{h} = \frac{((3+h)-1)^2-((3)-1)^2}{h} = \frac{(h+2)^2-(2)^2}{h} = \frac{h^2+4h+4-4}{h} = \frac{h^2+4h}{h}$$

$$= \boxed{h+4}$$

4. Find the instantaneous rate of change of $s(t) = t^3$ at $t = 3$.

$$\underset{t=3}{\text{IROC}} = \lim_{h\to 0}\frac{s(3+h)-s(3)}{h} = \lim_{h\to 0}\frac{(3+h)^3-3^3}{h} = \lim_{h\to 0}\frac{h^3+9h^2+27h+27-27}{h}$$

$$= \lim_{h\to 0}\frac{h^3+9h^2+27h}{h} = \lim_{h\to 0} h^2+9h+27 = \boxed{27}$$

For the remaining problems, you may use the following shortcut formula:

If $p(x) = ax^2 + bx + c$, *then* $p'(x) = 2ax + b$.

5. If $k(s) = 3s^2 + 2s - 2$, what is the value of $k(s)$ for which the instantaneous rate of change equals 8?

Since $k(s)$ *is a quadratic equation, we know that* $k'(s) = 6s + 2$. *If the instantaneous rate of change equals* 8, *then* $k'(s) = 6s + 2 = 8$. *Solving for* s, *we get* $s = 1$. *Then, the value of* $k(s)$ *when the instantaneous rate of change equals* 8 *is* $k(1) = 3(1)^2 + 2(1) - 2 = 3 + 2 - 2 = \boxed{3}$.

6. Let $h(x) = x^2 - 4x + 5$. Find a value of c in the interval $(1, 10)$ where the average rate of change of $h(x)$ from $x = 1$ to $x = 10$ is equal to the instantaneous rate of change of $h(x)$ at $x = c$.

We want a value of c *such that the IROC at* c *is equal to the AROC from 1 to 10. First,* $h(x)$ *is a quadratic equation, so* $h'(x) = 2x - 4$. *Now,*

$$\underset{1\to 10}{\text{AROC}} = \frac{f(10) - f(1)}{10 - 1} = \frac{((10)^2 - 4(10) + 5) - ((1)^2 - 4(1) + 5)}{10 - 1} = \frac{65 - 2}{10 - 1} = 7, \textit{ and}$$

$$\underset{x=c}{\text{IROC}} = h'(c) = 2c - 4. \textit{ So, we want } c \textit{ so that } 2c - 4 = 7. \textit{ Solving for } c\textit{, we get } c = \boxed{\frac{11}{2}}.$$

7. Find an equation of the tangent line to the graph of $f(x) = 7x^2 - 4x + 3$ at $x = -2$.

We know that the general form for the equation of a tangent line is $y = f(c) + f'(c)(x - c)$. *Since* $c = -2$ *for this problem, we have that* $f(c) = f(-2) = 7(-2)^2 - 4(-2) + 3 = 28 + 8 + 3 = 39$. *Also,* $f(x)$ *is a quadratic equation, so* $f'(x) = 14x - 4$. *Then,* $f'(c) = f'(-2) = 14(-2) - 4 = -28 - 4 = -32$. *Putting this all together, we have that the equation of the tangent line in question is* $\boxed{y = 39 - 32(x + 2)}$.

8. Suppose $R(x) = -3x^2 - x + 2$. What is the value of c such that the tangent line to $R(x)$ at c is perpendicular to the line $y = -2t - 3$?

We know that the derivative at c *is equal to the slope of the tangent line at* c. *Since our tangent line is to be perpendicular to the line* $y = -2t - 3$, *we need the slope of our tangent line (i.e. our derivative) to be* $\frac{1}{2}$. $R(x)$ *is a quadratic equation, so* $R'(x) = -6x - 1$. *So, we need a value of* c *such that* $R'(c) = -6c - 1 = \frac{1}{2}$. *Solving for* c, *we get* $c = \boxed{-\frac{1}{4}}$.

9. An object is launched up in the air. The height of the object after t seconds is $H(t)$ feet, where $H(t) = -16t^2 + 176t + 16$.

a.) When is the object at its greatest height? (Hint: What is the velocity at the object's greatest height?)

The object is at its greatest height when the velocity is 0. Since the velocity function is merely the derivative of the position function, we have the the velocity function is $H'(t) = -32t + 176$. *Now,* $H'(t) = -32t + 176 = 0$ *implies that* $t = \boxed{5.5 \textit{ seconds}}$.

b.) What is the maximum height of the object?

Since the object is at its greatest height when $t = 5.5$*, the maximum height must be* $H(5.5) = -16(5.5)^2 + 176(5.5) + 16 = -16(30.25) + 176(5.5) + 16 = -484 + 968 + 16 = \boxed{500 \textit{ feet}}$.

10. The displacement (in feet) of a certain particle moving in a straight line is given by $s(t) = \frac{1}{2}t^3 - t^2$, where t is measured in seconds.

a.) Find the average velocity over the following time periods. (2 pts.)

(i) $[1, 3]$ $\underset{1\to3}{\text{AROC}} = \dfrac{f(3) - f(1)}{3 - 1} = \dfrac{(\frac{1}{2}(3)^3 - (3)^2) - (\frac{1}{2}(1)^3 - (1)^2)}{3 - 1} = \dfrac{4.5 + 0.5}{3 - 1} = \boxed{2.5}$

(ii) $[1, 2]$ $\underset{1\to2}{\text{AROC}} = \dfrac{f(2) - f(1)}{2 - 1} = \dfrac{(\frac{1}{2}(2)^3 - (2)^2) - f(1)}{2 - 1} = \dfrac{0 + 0.5}{2 - 1} = \boxed{0.5}$

(iii) $[1, 1.5]$ $\underset{1\to1.5}{\text{AROC}} = \dfrac{f(1.5) - f(1)}{1.5 - 1} = \dfrac{(\frac{1}{2}(1.5)^3 - (1.5)^2) - f(1)}{1.5 - 1} = \dfrac{-.5625 + .5}{1.5 - 1} = \boxed{-0.125}$

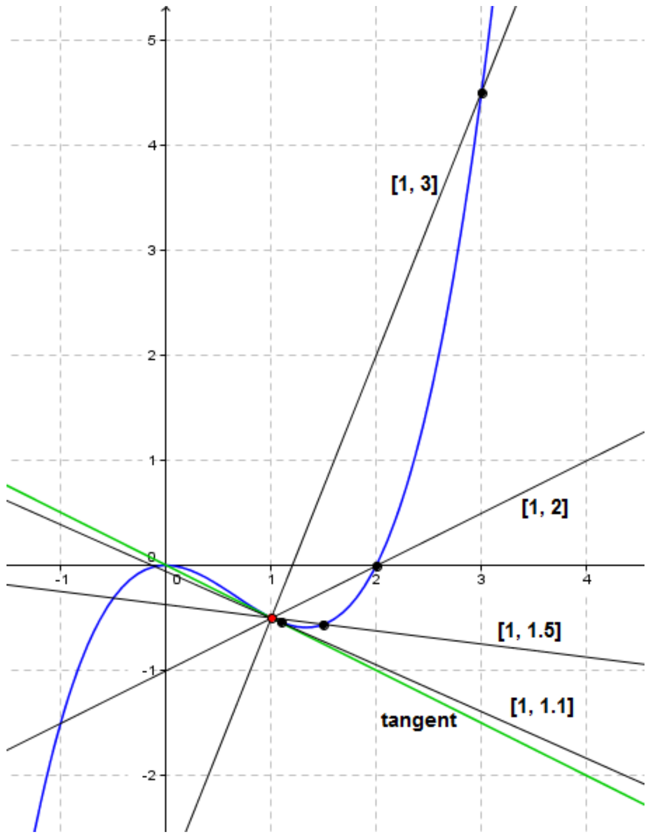
(iv) $[1, 1.1]$ $\underset{1\to1.1}{\text{AROC}} = \dfrac{f(1.1) - f(1)}{1.1 - 1} = \dfrac{(\frac{1}{2}(1.1)^3 - (1.1)^2) - f(1)}{1.1 - 1} = \dfrac{-.5445 + .5}{1.1 - 1} = \boxed{-0.445}$

b.) Find the instantaneous velocity when $t = 1$. (1 pt.)

$$
\begin{aligned}
\underset{t=1}{\text{IROC}} &= \lim_{h\to0} \frac{f(1+h) - f(1)}{h} \\
&= \lim_{h\to0} \frac{(\frac{1}{2}(1+h)^3 - (1+h)^2) - (\frac{1}{2}(1)^3 - (1)^2)}{h} \\
&= \lim_{h\to0} \frac{\frac{1}{2}(h^3 + 3h^2 + 3h + 1) - (h^2 + 2h + 1) + 0.5}{h} \\
&= \lim_{h\to0} \frac{\frac{1}{2}h^3 + \frac{3}{2}h^2 + \frac{3}{2}h + \frac{1}{2} - h^2 - 2h - 1 + 0.5}{h} \\
&= \lim_{h\to0} \frac{\frac{1}{2}h^3 - \frac{1}{2}h^2 - \frac{1}{2}h}{h} \\
&= \lim_{h\to0} \frac{1}{2}h^2 - \frac{1}{2}h - \frac{1}{2} \\
&= \boxed{-\frac{1}{2}}
\end{aligned}
$$

c.) Draw the graph of s as a function of t, and draw the secant lines whose slopes are the average velocities found in part (a). (2 pts.)

The graph of s is the blue curve in the graph below. The secant lines are drawn in black with their corresponding intervals.



d.) Draw the tangent line whose slope is the instantaneous velocity from part (b). (1 pt.)

The tangent line is the green line in the graph above.